Since det $T_\theta \neq 0$, $\tilde{\lambda}_k \neq 0$. From these relations, we have

$$\det(\tilde{T}_\theta - \tilde{\lambda}_k I)\det(\Omega^{-1}T_\theta) = \det(\tilde{T}_\theta\Omega^{-1}T_\theta - \tilde{\lambda}_k\Omega^{-1}T_\theta)$$
$$= \det(\Omega^{-1} - \tilde{\lambda}_k\Omega^{-1}T_\theta)$$
$$= \det(\Omega^{-1}\tilde{\lambda}_k)\det\left(\frac{1}{\tilde{\lambda}_k}I - T_\theta\right) = 0.$$

The final result is

$$\det\left(T_\theta - \frac{1}{\tilde{\lambda}_k}I\right) = 0. \quad (53)$$

From (53), it follows that there is always an eigenvalue $1/\tilde{\lambda}_k$ of T_θ, corresponding to an eigenvalue $\tilde{\lambda}_k$ to $\tilde{T}_\theta$, (Theorem 1).

We shall use the same subscript for the corresponding solutions of the eigenvalue problems of the two circuits:

$$\lambda_k = \frac{1}{\tilde{\lambda}_k}. \quad (54)$$

Multiplying (21) by $\Omega^{-1}T_\theta A_l$ from the right and using (19) and (20), we obtain

$$\left(\lambda_l - \frac{1}{\tilde{\lambda}_k}\right)\tilde{A}_k\Omega^{-1}A_l = 0. \quad (55)$$

If $\lambda_l \neq 1/\tilde{\lambda}_k$, (55) shows that $\tilde{A}_k\Omega^{-1}A_l = 0$. In the non-degenerate case, $\lambda_l \neq 1/\tilde{\lambda}_k$ for $k \neq l$. Thus, we obtain the desired orthogonality relation (Theorem 2):

$$\tilde{A}_k\Omega^{-1}A_l = 0, \qquad k \neq l. \quad (56)$$

In the degenerate case, $k \neq l$ does not necessarily mean that $\lambda_l \neq 1/\tilde{\lambda}_k$. However, we are justified in assuming (56), for it is always possible to introduce the degenerate eigenvectors in such a way as to secure the orthogonality.

Next, we expand $\Omega\tilde{A}_k^+$ by the eigenvectors A_l, where the symbol $^+$ indicates the complex conjugate transpose:

$$\Omega\tilde{A}_k^+ = \sum \alpha_l A_l.$$

Multiplying by $\tilde{A}_k\Omega^{-1}$ from the left and using (56), we have

$$\tilde{A}_k\tilde{A}_k^+ = \alpha_k\tilde{A}_k\Omega^{-1}A_k.$$

Since $\tilde{A}_k \neq 0$, the left hand side of the above equation is not zero. Thus we conclude that (Theorem 3):

$$\tilde{A}_k\Omega^{-1}A_k \neq 0. \quad (57)$$

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Action of a Progressive Disturbance on a Guided Electromagnetic Wave*

J. C. SIMON†

I. INTRODUCTION

A.

A PROBLEM often encountered in wave physics concerns the interaction of various types of waves, and the energy transfer from one wave to another.

In the particular case of waves of the same nature, "modes" can be distinguished in such a way that a wave can be represented as a sum of these modes. Their essential character is that the energy associated with each does not vary with time. It is also said that these modes are not "coupled." This, for instance, is the case of waves guided in an electric waveguide, of mechanical vibration in a bar, and of energy levels in quantum physics.

Although this possibility of decomposition in "normal modes" corresponds to particular physical conditions, it has made it possible to deduce general notions of a fundamental character essential to the physicist. In the most general case, the normal modes are said to be coupled that is the energy passes from one to the other, so much that this decomposition into normal modes appears to be indispensable in deducing physical concepts.

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Such problems are treated by using a method of approximation known as the "theory of perturbations." Much could no doubt be said about the validity of the application of this method and the convergence of solutions. Nevertheless, it is frequently employed in many fields of physics, in particular, in quantum physics such as solid-state physics or atomic physics. It has enabled physicists to obtain results which have been confirmed by experiment. We shall therefore apply it to the particular problem of the action of a progressive disturbance on an electromagnetic wave.

B.

One of the principal applications of the study of the action of a progressive disturbance will, as we shall see, concern parametric amplification.

What are the essentials of parametric amplification? They consist of a signal to be amplified, of frequency $\omega/2\pi$, "pumping" energy at frequency $\omega_1/2\pi$, and a medium whose characteristics vary in function with the applied pumping energy. Usually, "pumping" energy and signal energy are of the same kind, electromagnetic for instance. It is always implied that it is the "pumping" energy alone which acts on the medium, to the exclusion of the signal or of the resulting beats.

Therefore, it appears legitimate to say that *it is the medium modified by the pumping which acts on the signal.* Pumping can therefore be ignored in formulating the problem which in any case becomes much clearer physically.

A modification of the medium may be produced by something other than an electromagnetic wave—by a mechanical wave, for instance, as in the case of heat photons and X-rays.

Thus, the following scheme may be adopted. Because of its energy, the pumping modifies the medium (ϵ or μ variable as a function of the pumping field). Knowing the modification of the medium, an action on the signal can be deduced. *This point of view is, of course, legitimate only because the Maxwell equations are linear for the signal, which is assumed not to act on the medium* (for small signals approximation, see Section V, B).

Modification of a medium can be obtained in various ways. In the case of electromagnetic pumping energy, it is naturally necessary that the characteristics ϵ or μ vary with the level of the field. *The medium is said to be nonlinear.* This is obtained in general only for rather high pumping energy, or, in any case, energy much greater than that of the incident signal.

Action on a nonlinear medium of an electromagnetic field in order to modify appreciably the characteristics of the medium is a difficult problem. It must be dealt with if the problems of parametric amplification are to be fully solved.

However, in the case of a progressive pumping wave, it appears physically plausible that the modification of the medium is akin to a sinusoidal disturbance accompanying the pumping wave, at least as a first approximation. For this reason the disturbance of the medium will be described by the relations (1) or (1′). It should be noted that such a disturbance can arise only if the medium "follows" the electromagnetic field at the frequency of the pumping wave. This condition limits parametric amplification at the higher frequencies.

Section II

A. Establishing the General Propagation Equation

Consider a three dimensional medium, such that $\mu=\mu_0=c^{te}$, $\epsilon_0=c^{te}$:

$$\epsilon = \epsilon_0 + \epsilon_1 \cos(\omega_1 t - \bar{k}_1\cdot\bar{r}). \quad (1)$$

The components of $\bar{r}$ are the direction cosines of direction $\bar{k}_1(0, 0, k_1)$.

The Maxwell equations are written:

$$\nabla\times\overline{E} = -\frac{\partial\overline{B}}{\partial t} \quad (2) \qquad \nabla\times\overline{H} = \frac{\partial\overline{D}}{\partial t} \quad (4)$$

$$\nabla\cdot\overline{B} = 0 \quad (3) \qquad \nabla\cdot\overline{D} = 0. \quad (5)$$

Eliminating $\overline{H}$ and $\overline{B}=\mu_0\overline{H}$ from (2) and (4), we have

$$\nabla\times\nabla\times\overline{E} = -\mu_0\frac{\partial^2\overline{D}}{\partial t^2} = \nabla(\nabla\cdot\overline{E}) - \nabla^2\overline{E}. \quad (6)$$

Assume $E_z=0$. As ϵ does not vary following directions x and y, (5) becomes $\nabla\cdot\overline{E}=0$; under these conditions (6) takes the form

$$\nabla^2\overline{E} = \mu_0\frac{\partial^2\overline{D}}{\partial t^2}\,. \quad (7)$$

It should be noted that formulas corresponding to a variable permeability are written in similar fashion if similar hypotheses can be made on μ. Let

$$\epsilon = \epsilon_0 = \text{constant and } \mu = \mu_0 + \mu_1\cos(\omega_1 t - \bar{k}_1\bar{r}). \quad (1')$$

Taking $H_z=0$ we have

$$\nabla^2\overline{H} = \epsilon_0\frac{\partial^2\overline{B}}{\partial t^2}\,. \quad (7')$$

B. Introduction of Boundary Conditions

The conditions in cases where one of the two parameters ϵ or μ is variable are satisfied by TEM modes guided in the direction O_z. This, in particular, is the case if the guiding structure consists of two plane walls of zero impedance, and of two perpendicular walls of infinite impedance. A portion of a plane wave can be propagated in such a guided structure. In the case of the usual waveguide with zero impedance walls, it is the variable ϵ case, magnetic mode, which satisfies simply the boundary conditions. Let us deal with this case, from which the preceding case is easily deduced.

Take a metallic waveguide with sides a and b. The magnetic modes satisfy $E_z=0$ and $\nabla\cdot\bar{E}=0$ (see Fig. 1). Let H_{01} be the fundamental mode

$$\left[\begin{array}{l} E_x = E_0 \sin \frac{\pi y}{b} P \\ E_y = 0 \\ E_z = 0 \end{array}\right. \quad (8) \qquad \left[\begin{array}{l} H_x = 0 \\ \mu_0 \frac{\partial H_y}{\partial t} = -\frac{\partial E_x}{\partial z} \\ \mu_0 \frac{\partial H_z}{\partial t} = -\frac{\partial E_x}{\partial y} \end{array}\right. \quad (9)$$

with $P=\exp -j(\omega t-kz)$.

Eqs. (8) and (9) are valid when the waveguide is filled with a homogeneous material. In order to make it valid in the case where ϵ satisfies (1), we write:

$$E_x = E_0 \sin \frac{\pi y}{b} P\cdot \sum_{-\infty}^{+\infty} a_n \exp - jn(\omega_1 t - k_1 z). \quad (8')$$

Eqs. (2) and (4) are satisfied if (6) is satisfied.

Since (5) is satisfied because of the choice of $\bar{E}$, (6) takes the form of (7). If (7) is satisfied, $\bar{H}$ is deduced from $\bar{E}$ by group (9), which is deduced from (2). It is easy to verify that the boundary conditions are also satisfied.

The term under the exponential is written:

$$(\omega + n\omega_1)t - (k + nk_1)z.$$

If ω_1 is of the order of ω, rigorously speaking, it would be necessary to introduce modes of higher order, H_{0p}, since the latter could be propagated.

Let us restrict ourselves to the case of a sum of modes H_{01}. Transfer (8′) into (7), which is written:

$$\frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = \mu_0 \frac{\partial t^2}{\partial t^2} \left\{ [\epsilon_0 + \epsilon_1 \cos (\omega_1 t - k_1 z)] E_x \right\}, \quad (7)$$

and transform the cosine into an exponential sum, ordinating in n. The resulting equation will be satisfied if the coefficients of the variable terms are all zero.

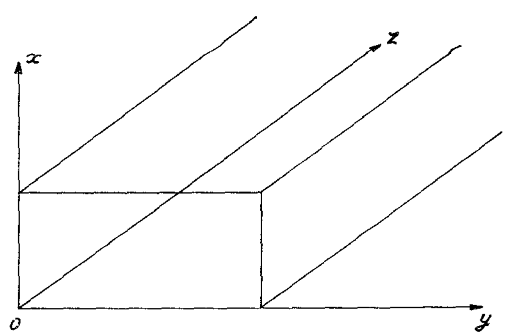

Fig. 1

For this we must have

$$\left[\epsilon_0 - \frac{(k + nk_1)^2 + \frac{\pi^2}{b^2}}{\mu_0(\omega + n\omega_1)^2} \right] a_n + \frac{\epsilon_1}{2}(a_{n-1} + a_{n+1}) = 0. \quad (10)$$

Verification: make $\epsilon_1=0$. We have only one coefficient $a_n\neq 0$, *i.e.*, a_0, if

$$\omega^2\epsilon_0\mu_0 - \frac{\pi^2}{b^2} - k^2 = 0, \quad (11)$$

but

$$\omega^2\epsilon_0\mu_0 = k_0^2 \quad \left(k_0 = \frac{2\pi}{\lambda_0}; \lambda_0 = \text{wavelength in vacuum}\right)$$

and (11) becomes

$$k_0^2 - k^2 = \frac{\pi^2}{b^2}. \quad (11')$$

C. The Perturbation Method

The solution of (7) has become the solution of a system of an infinity of homogeneous equations with an infinite number of unknowns. Such a process is often employed in mathematical physics. It is the one used, for instance, in solid-state physics,[1] or in quantum physics when a solution is sought for the perturbed Schrödinger equation.[2] Historically, astronomers Mathieu and Hill were the first to use such a mathematical technique.[3]

The system whose general equation is given by (10) has a solution only if the determinant is zero. We then have to find the values of ω and k which make an infinite determinant zero. The general problem is very complex, so we shall only introduce approximations which will give the result simply.

Examination of (10) shows that the a_n coefficient bracket is large compared to $\epsilon_1/2$ coefficient of the term $a_{n-1}+a_{n+1}$. It is desirable to obtain an expression for a_n in which ϵ_1 may be considered as being infinitely small. This is the "perturbation method."

For instance, let us try to solve the system step by step, taking two a_n, a_0 and a_1. We establish that a_n tends to infinity. It is possible that a_n tends to zero for n infinite positive or infinite negative, but not for both. This, of course, is due to the fact that the system determinant is not zero.

Assume a_n infinitely small compared to a_0 in $\epsilon_1^{|n|}$ and ignore infinitely small terms of an order greater than 2.

[1] See section 40 of [2].
[2] See section 2 of [1].
[3] See Chapter 19 of [3].

We shall then say that we are using the perturbation method of order 2. From the physics point of view, this means ignoring beats of order greater than $+1$ and -1.

Only three equations of the type of (10) are involved; those corresponding to $n=-1$, 0, and $+1$. That is to say

$$[n=-1]a_{-1}+\frac{\epsilon_1}{2}a_0=0 \tag{12}$$

$$[n=0]a_0+\frac{\epsilon_1}{2}(a_{-1}+a_{+1})=0 \tag{13}$$

$$[n=+1]a_1+\frac{\epsilon_1}{2}a_0=0. \tag{14}$$

Making these three equations compatible, we have

$$[n=0]-\frac{\epsilon_1^2}{4}\left[\frac{1}{[n=+1]}+\frac{1}{[n=-1]}\right]=0. \tag{15}$$

Eq. (15) connects ω and k as a function of parameters ω_1, k_1 and ϵ_1/ϵ_0.

In order to simplify the discussion without changing the physical conclusions, let us restrict ourselves to the case of the ideal TEM mode, already mentioned—that of a waveguide which has two walls of zero impedance and two of infinite impedance. All that is needed is to write in (10) $1/b=0$. Eq. (15) is written, remembering that $k_0^2=\epsilon_0\mu_0\omega^2$,

$$\left(1-\frac{k^2}{k_0^2}\right)-\frac{\epsilon_1^2}{4\epsilon_0^2}\cdot\left[\frac{1}{1-\dfrac{(k+k_1)^2}{k_0^2\left(1+\dfrac{\omega_1}{\omega}\right)^2}}+\frac{1}{1-\dfrac{(k-k_1)^2}{k_0^2\left(1-\dfrac{\omega_1}{\omega}\right)^2}}\right]=0.$$

Let

$$\frac{k}{k_0}=X;\quad \frac{k_1}{k_0}=X_1;\quad \frac{\epsilon_1^2}{4\epsilon_0^2}=Z;\quad \frac{\omega_1}{\omega}=\Omega.$$

We have

$$\frac{1}{Z}=\frac{1}{1-X^2}\left[\frac{1}{1-\left(\dfrac{X+X_1}{1+\Omega}\right)^2}+\frac{1}{1-\left(\dfrac{X-X_1}{1-\Omega}\right)^2}\right]. \tag{16}$$

D. The Various Solutions

Because of the approximations of the perturbation theory, (16) may give correct results only for small and obviously positive values of Z, and for $|X|\sim 1$. It should be noted that (16), $Z=f(X)$, does not change if X and X_1 change to $-X$ and $-X_1$. This only means changing the sense of the axis Oz, thus changing nothing in the physical conditions. From this, it is possible to restrict the study of the approximation $Z=f(X)$ near the point $X=+1$, $Z=0$.

Eq. $f_x=0$ is satisfied for six values of X:

$$\pm 1;\qquad -X_1\pm(1+\Omega);\qquad +X_1\pm(1-\Omega).$$

In general, around $X=1$ there is a real solution, and one only, to $Z_0=f(X)$. This solution is real and little different from unity; the value of k is real and little different from k_0. This is no longer the case if one of the preceding roots is close to unity. Complex solutions of $Z_0=f(X)$ in X can appear. This will naturally happen only in the presence of double or triple roots. Let us examine the various possible cases:

1) $1=-X_1+1+\Omega$ $\quad X_1=\Omega$ triple root (see 3)
2) $1=-X_1-1-\Omega$ $\quad X_1=-2-\Omega$ double root
3) $1=X_1+1-\Omega$ $\quad X_1=\Omega$ triple root (see 1)
4) $1=X_1-1+\Omega$ $\quad X_1=2-\Omega$ double root.

It is easy to prove that the solutions corresponding to point $(-1, 0)$ are deduced from the latter by changing X_1 to $-X_1$; that is, by a simple change of the orientation of the axis Oz (see Fig. 2).

We shall therefore examine the case of triple roots 1) and 3), and the cases of double roots 2) and 4).

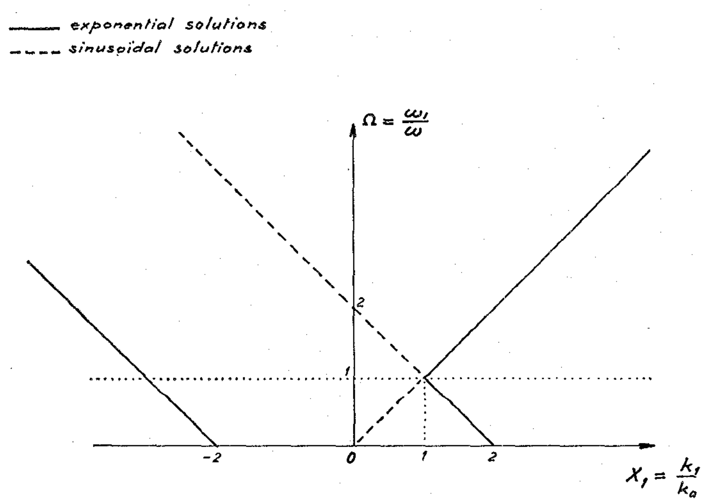

Fig. 2

E. Remarks

The relation 1) or 3) can be written

$$\frac{k_1}{\omega_1} = \frac{k_0}{\omega} = \frac{k_1 + k_0}{\omega_1 + \omega} = \frac{k_0 - k_1}{\omega - \omega_1}.$$

The field formula (8′) shows that it is written, in general, in the form of the sum of three waves corresponding to $n = -1$, $n = 0$, $n = +1$.

If ϵ_1 is very small compared to ϵ_0, k is very little different from k_0, to within the perturbation term of k, the three waves in question have phases respectively equal to

$$(\omega - \omega_1)t - (k_0 - k_1)z;\ \omega t - k_0 z;\ (\omega + \omega_1)t - (k_0 + k_1)z.$$

Relation 1) therefore means that *these three waves have equal phase velocities.*

Similarly, the relations 2) and 4) are written:

$$\frac{k_1 + k_0}{\omega_1 + \omega} = -\frac{k_0}{\omega} \tag{2a}$$

$$\frac{k_1 - k_0}{\omega_1 - \omega} = -\frac{k_0}{\omega}. \tag{4a}$$

In these cases, waves $n = 0$ and $n = +1$ or $n = -1$ have phase velocities which have *equal×absolute value but of opposite signs.* Physicists say that this concerns the Bragg phenomenon, and, in the cases considered previously, the *Bragg interference of the first order.*

Thus, the general formula for the Bragg phenomenon corresponds to the formula

$$\left(\frac{k_0 + nk_1}{\omega + n\omega_1}\right)^2 = \left(\frac{k_0}{\omega}\right)^2;$$

n is the order of the interference. Physically, this means that the wave with the phase factor $(\omega + n\omega_1)t - (k_0 + nk_1)z$ and the wave $\omega t - k_0 z$ which correspond respectively to the terms n and 0 in the development of (8′), have phase velocities which have equal×absolute value. From the calculation point of view, the consequence of the above relation is that among (12), (13) and (14), equations corresponding to ranks n and 0 have equal coefficients. *The system is degenerate.*

Section III

A. Double Root Cases

In this case, one of the brackets of (12) or (14) is cancelled. It is no longer possible to say that the corresponding value of a_{+1} or a_{-1} is small compared to the value of a_0. If, for instance, it is the bracket corresponding to $n = +1$ which becomes zero when X is equal to unity, a_{+1} is of the order of a_0, but then a_{-1} is of the order of ϵ_1, and is therefore negligible compared to a_0 and a_1. Eqs. (12), (13) and (14) are reduced to (13) and (14) in which a_{-1} has been made zero. In quantum physics this solution is termed degenerate. We have seen that it corresponds to the Bragg interference case.

B. Case $X_1 = -2 - \Omega$

It is the bracket corresponding to $n = +1$ which becomes zero if X tends towards unity. We shall assume that a_{+1} is of the order of a_0 and that a_{-1} is negligible. Eqs. (13) and (14) alone are to be considered and are written:

$$[n = 0]a_0 + \frac{\epsilon_1}{2} a_1 = 0 \tag{13′}$$

$$[n = +1]a_1 + \frac{\epsilon_1}{2} a_0 = 0. \tag{14′}$$

In order that they shall be compatible, we must have

$$[n = 1][n = 0] - \frac{\epsilon_1^2}{4} = 0. \tag{15′}$$

Using the notation previously adopted,

$$\frac{k}{k_0} = X; \qquad \frac{k_1}{k_0} = X_1; \qquad \frac{\epsilon_1^2}{4\epsilon_0^2} = Z; \qquad \frac{\omega_1}{\omega} = \Omega;$$

we have

$$Z = (1 - X^2)\left[1 - \left(\frac{X + X_1}{1 + \Omega}\right)^2\right].$$

Replacing X_1 by its value and neglecting infinitely small terms of order 2 and beyond, we have with $\alpha = \pm 1$; $j^2 = -1$,

$$\frac{k}{k_0} = 1 + \alpha j \frac{\epsilon_1}{4\epsilon_0}\sqrt{1 + \frac{\omega_1}{\omega}}. \tag{17}$$

Let us calculate the corresponding values of the field. Inserting the value of X in either (13′) or (14′), ϵ_1 disappears. This justifies the hypothesis that a_0 and a_1 are of the same order, and we have

$$\frac{a_1}{a_0} = +\alpha j\sqrt{1 + \frac{\omega_1}{\omega}}. \tag{18}$$

Finally, the value of the electric field can be written in the form:

$$E_x = a \exp - \alpha k_0 z \frac{\epsilon_1}{4\epsilon_0}\sqrt{1 + \frac{\omega_1}{\omega}}\left[\exp - j(\omega t - k_0 z) + \alpha j\sqrt{1 + \frac{\omega_1}{\omega}} \exp - j[(\omega + \omega_1)t - (k_1 + k_0)z]\right]. \tag{19}$$

It can be easily verified that the result is not fundamentally changed by a change of phase on the wave of phase $\omega t - k_0 z$ and therefore equally on $(\omega_1 t - k_1 z)$. Terms in ω and $\omega + \omega_1$ are still in quadrature and are given by (18), the ratio k/k_0 being given by (17). Let us write

$$\sqrt{1 + \frac{\omega_1}{\omega}} = b; \qquad \frac{\epsilon_1}{4\epsilon_0}\sqrt{1 + \frac{\omega_1}{\omega}} = \mu.$$

We have:

$$E_x = a \exp - \alpha\mu z k_0[\cos(\omega t - k_0 z) + \alpha b \sin[(\omega + \omega_1)t - (k_0 + k_1)z]]. \tag{20}$$

$$Hy = \frac{ak_0}{\mu_0\omega} \exp - \alpha\mu z k_0[\cos(\omega t - k_0 z) + \alpha b \sin[(\omega + \omega_1)t - (k_0 + k_1)z]]. \tag{21}$$

Eq. (21) is obtained by inserting (20) in the second part of (9), taking into account the relation (2a), which can be written

$$\frac{k_1 + k_0}{\omega_1 + \omega} = -\frac{k_0}{\omega},$$

and neglecting the term in ϵ_1/ϵ_0. The general expression for the field is written in the form of the sum of two terms corresponding to the values $\alpha = +1$ and $\alpha = -1$, each having a coefficient which, as we shall see, depends on the boundary conditions. For instance:

$$E_x = a_1\phi_{+1}(z) \exp - \mu k_0 z + a_2\phi_{-1}(z) \exp \mu k_0 z. \tag{20'}$$

$\phi(z)$, equal to the bracket in (20), in which α has been made equal to $+1$ or to -1, is a periodic function of z_0. We find a general expression in accordance with Floquet's theorem [3].

Let us now try to adapt these solutions to a non-perturbed medium. First, it should be noted that (20) and (21) show that the solution comprises two waves circulating in opposite directions, one of frequency ω in the positive sense, the other of frequency $\omega+\omega_1$ in the negative sense.

Let $\epsilon = \epsilon_0$ everywhere except in the segment $OA = z_0$, where it satisfies (1). The incident wave is the wave P_ω, such that $E_x = a \cos(\omega t - k_0 z)$. In segment OA only two groups of waves can exist. Each one of these groups consists of two waves, circulating in opposite directions, of frequencies $\omega/2\pi$ and $\omega+\omega_1/2\pi$. One decreases, and the other increases exponentially with z. They must come into accord in planes O and A. Because of the direction of propagation, it is possible to add only a wave $P_{\omega+\omega_1}$ of frequency $\omega+\omega_1/2\pi$ for $z<0$ and a wave P_ω'' of frequency $\omega/2\pi$ for $z>z_0$. In order to satisfy the boundary conditions the wave $P_{\omega+\omega_1}$ must become zero for $z=z_0$.

Neglecting terms in ϵ_1/ϵ_0, the boundary conditions are easily satisfied in the general case and make it possible to adopt the scheme of Fig. 3. One case of particular interest occurs when z is sufficiently large so that $\mu k_0 z_0$ is large. In this case P_ω'' is negligible, and it is only necessary to consider the group of waves with a negative exponential in order to satisfy the boundary conditions. By equating the electric and magnetic fields at the right and left of plane $z=0$, we have $a=a_1$. And if P_ω and $P_{\omega+\omega_1}$ represent the powers of the incident and reflected waves, we have

$$\frac{P_\omega}{\omega} = \frac{P_{\omega+\omega_1}}{\omega + \omega_1}. \tag{22}$$

The reflected wave is of frequency $\omega+\omega_1/2\pi$ and (22) shows that the ratio of the reflected to the incident waves is in the ratio of the frequencies.

The physical interpretation is simple. Let $\omega_1=0$, and we have $k_1=2k_0$. The medium is modulated sinusoidally at the spatial period of $\lambda_0/2$. It is well known that in this case the incident wave is reflected if the disturbance is large enough, and that in the disturbed medium the field is represented by two equal waves propagated in opposite directions and damped exponentially. Physicists designate this a case of *Bragg interference of the first order*, while for filter specialists it is a case of *a stopped band*.

When ω_1 differs from zero, things happen as if the medium moved toward negative values of z (in Fig. 3 wave P_1). The reflected wave is now at the upper frequency $\omega+\omega_1/2\pi$. The incident wave appears to be reflected by a moving mirror. This is a *Doppler effect*.

Finally, (22) is familiar to quantum physicists. It shows that to an incident photon of power $(h/2\pi)\omega$ there corresponds a reflected photon of power $h/2\pi(\omega+\omega_1)$. The number of photons is preserved.

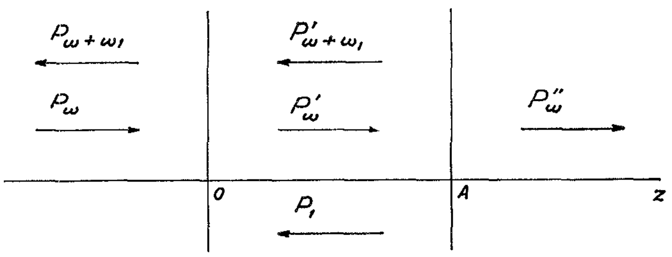

Fig. 3

C. Balance of Power

The reflected power is greater than the incident power. Energy has been transferred, obviously from the wave P_1 of the medium. Let us examine the balance of power in a slice dz with an abscissa less than z_0.

We can write:

$$\nabla\cdot(\overline{E} \times \overline{H}) = \overline{H}\cdot\nabla \times \overline{E} - \overline{E}\cdot\nabla \times \overline{H},$$

from which, using (2) and (4):

$$\nabla\cdot(\overline{E} \times \overline{H}) = -\overline{E}\cdot\frac{\partial\overline{D}}{\partial t} - \overline{H}\cdot\frac{\partial\overline{B}}{\partial t}. \tag{23}$$

Rigorously speaking, in order to establish (23) it would be necessary to consider the electric current density J[4].

[4] See section 2.19 of [4].

Term $\overline{E}\cdot\overline{J}$ would then appear in the first member of (23). But, in the case under consideration $\overline{J}$ is zero in the dielectric medium, and on the conducting walls the product $\overline{E}\cdot\overline{J}$ is obviously zero. Integrating over the volume V, we have

$$\int_S (\overline{E}\times\overline{H})_n\, dS = -\int_V \left(\overline{E}\frac{\partial\overline{D}}{\partial t} + \overline{H}\frac{\partial\overline{B}}{\partial t}\right) dV. \quad (24)$$

Let τ_1 represent the first member of (24), and τ_2 the second member. τ_2 is written

$$-\tau_2 = \frac{1}{2}\int_V \frac{\partial}{\partial t}(\epsilon E^2 + \mu H^2)dV + \frac{1}{2}\int_V E^2\frac{\partial\epsilon}{\partial t}dV. \quad (25)$$

Eq. (25) differs from the usual formula with ϵ constant because of the second integral. If the fields expressed by (20) and (21) are inserted in (25), and if the average value of τ_2 is calculated over a sufficiently long interval of time, we have

$$\bar{\tau}_2 = +\frac{\alpha}{4} b\omega_1\epsilon_1 E_0^2 dz,$$

which arises exclusively from the second integral of (25). Further,

$$\tau_1 = \int_S E_0^2 \frac{k}{\mu_0\omega} \exp - 2\alpha\mu z k_0$$
$$\cdot\left[\cos^2(\omega t - kz) - \alpha^2 b^2 \sin^2\left[(\omega+\omega_1)t - (k+k_1)z\right]\right].$$

Taking the average value of τ_1 we have

$$\bar{\tau}_1 = dzE_0^2\cdot 2\alpha\mu k_0\cdot\frac{k}{\mu_0\omega}\frac{1}{2}[b^2 - 1], \quad (26)$$

but

$$b = \sqrt{1 + \frac{\omega_1}{\omega}}; \qquad k^2 = \epsilon_0\mu_0\omega^2;$$

$$\mu = \frac{\epsilon_1}{4\epsilon_0}\times\sqrt{1+\frac{\omega_1}{\omega}},$$

and naturally, $\bar{\tau}_1 = \bar{\tau}_2$. This verification shows that the fields actually satisfy the equations and also that τ, given by

$$\tau = \frac{1}{2}\int_V E^2\frac{\partial\epsilon}{\partial t}dV, \quad (27)$$

represents the work done by the medium, τ_1 being equal to the flux of the Poynting vector, or to the energy carried away by the electromagnetic wave. Eq. (26) shows that *this energy is proportional to* ϵ_1/ϵ_0, ω_1 *and to the incident energy.*

We have implicitly assumed that the medium was capable of supplying energy without becoming modified. Naturally this is only an approximation, all the closer to reality as the quantity of energy is small. This is the case of weak incident energy—*a case of "approximation for small signals."*

D. Case $X_1 = 2 - \Omega$

The calculations are similar to those for the previous case. But now it is the bracket corresponding to $n = -1$ which becomes zero if X tends towards unity. Eqs. (12) and (13) alone are to be considered, a_{+1} being negligible. The equation which gives k in terms of the other parameters is written

$$Z = (1 - X^2)\left[1 - \left(\frac{X - X_1}{1 - \Omega}\right)^2\right]. \quad (28)$$

Replacing X_1 by its value, we have for $\alpha = \pm 1$:

$$\frac{k}{k_0} = 1 + \alpha j\frac{\epsilon_1}{4\epsilon_0}\sqrt{1 - \frac{\omega_1}{\omega}} \quad \text{if } \omega_1 < \omega \quad (29)$$

$$\frac{k}{k_0} = 1 + \alpha\frac{\epsilon_1}{4\epsilon_0}\sqrt{\frac{\omega_1}{\omega} - 1} \quad \text{if } \omega_1 > \omega. \quad (30)$$

The solutions corresponding to $\omega_1 < \omega_0$ or $\omega_1 > \omega_0$ are now of a different kind; one is exponential, the other purely sinusoidal. The ratio a_{-1}/a_0 is written:

$$\frac{a_{-1}}{a_0} = \alpha j\sqrt{1 - \frac{\omega_1}{\omega}} \quad \text{if } \omega_1 < \omega \quad (31)$$

$$\frac{a_{-1}}{a_0} = \alpha\sqrt{\frac{\omega_1}{\omega} - 1} \quad \text{if } \omega_1 > \omega. \quad (32)$$

E. $\omega_1 < \omega$

The electric and magnetic fields, for

$$\sqrt{1 - \frac{\omega_1}{\omega}} = b; \qquad \frac{\epsilon_1}{4\epsilon_0}\sqrt{1 - \frac{\omega_1}{\omega}} = \mu,$$

where

$$\frac{k_1 - k_0}{\omega_1 - \omega} = -\frac{k_0}{\omega},$$

are written

$$E_x = a \exp - \alpha\mu z k_0\left[\cos(\omega t - k_0 z) + \alpha b \sin\left[(\omega - \omega_1)t - (k_0 - k_1)z\right]\right], \quad (33)$$

$$Hy = \frac{ak_0}{\mu_0\omega}\exp - \alpha\mu z k_0\left[\cos(\omega t - k_0 z) - \alpha b\sin\left[(\omega - \omega_1)t - (k_0 - k_1)z\right]\right]. \quad (34)$$

These expressions are very close to (20) and (21). Continuing with the reasoning in the previous paragraph, it is easily seen that the solution described by (33) and (34) consists of two waves P_ω and $P_{\omega-\omega_1}$ of frequency $\omega/2\pi$ and $\omega-\omega_1/2\pi$ circulating in opposite directions.

In the case of Fig. 4, if z_0 is sufficiently large, only wave $P_{\omega-\omega_1}$ at frequency $\omega-\omega_1/2\pi$, is reflected and the relation of the conservation of the number of photons is again satisfied:

$$\frac{P_\omega}{\omega} = \frac{P_{\omega-\omega_1}}{\omega-\omega_1}.$$

The perturbation wave P_1 of the medium moves in the positive direction. We are again dealing with Doppler reflection on a medium moving away instead of approaching, as in the previous case. The frequency as well as the energy decrease. Energy is imparted to the medium.

F. $\omega_1 > \omega$

The conditions of Fig. 4 are still valid. In the perturbed medium two groups of two waves of pulsation ω and $\omega_1-\omega$ can be propagated. A value of α corresponds to each one of these groups. For instance, the electric field is written

$$\begin{aligned} E_x = {} & [a_1 \exp jk_0\mu z + a_2 \exp - jk_0\mu z] \exp - j(\omega t - k_0 z) \\ & + b[a_1 \exp + jk_0\mu z - a_2 \exp - jk_0\mu z] \\ & \cdot \exp + j[(\omega_1 - \omega)t + (k_0 - k_1)z], \end{aligned} \quad (35)$$

where

$$b = \sqrt{\frac{\omega_1}{\omega} - 1} \quad \text{and} \quad \mu = \frac{\epsilon_1}{4\epsilon_0}\sqrt{\frac{\omega_1}{\omega} - 1}.$$

Writing that the boundary conditions are satisfied, in O and A we have

$$a_1 + a_2 = a, \quad (36)$$

$$a_1 \exp jk_0\mu z_0 - a_2 \exp - jk_0\mu z_0 = 0. \quad (37)$$

Hence,

$$a_1 = \frac{a}{2}[1 - jtgk_0\mu z_0]$$

$$a_2 = \frac{a}{2}[1 + jtgk_0\mu z_0].$$

Two cases are of special interest:

$$\frac{k_0\mu z_0 = +\kappa\pi}{a_1 = a_2 = \frac{a}{2}}.$$

Eq. (35) becomes

$$\begin{aligned} E_x = {} & a \cos k_0\mu z \cos(\omega t - k_0 z) \\ & - ab \sin k_0\mu z \sin[(\omega_1 - \omega)t + (k_0 - k_1)z]. \end{aligned} \quad (38)$$

For $z=0$ or $z=z_0$, the first term of (38) alone remains, the incident wave is transmitted unchanged, and no wave of pulsation $\omega_1-\omega$ issues.

$$k_0\mu z_0 = \frac{\pi}{2} + \kappa\pi.$$

System (36), (37) is degenerate, and there is no solution for it unless $a=0$. In this case with $a_1=a_2=a'/2$, (35) is written

$$\begin{aligned} E_x = {} & a' \sin k_0\mu z \sin(\omega t - k_0 z) \\ & + a'b \cos k_0\mu z \cos[(\omega_1 - \omega)t + (k_0 - k_1)z]. \end{aligned} \quad (39)$$

It is possible to reconcile this solution in O and A by the method shown in Fig. 5: a wave P_ω'' toward the right and a wave $P_{\omega_1-\omega}$ toward the left. It should be noted that the relation

$$\frac{P_\omega''}{\omega} = \frac{P_{\omega_1-\omega}}{\omega_1-\omega} \quad (40)$$

is verified, as is readily seen in (39).

Of course, it is still necessary to find out *how* such a solution can be established in the perturbed medium (note that a' is arbitrary. However, it can be asserted that the system oscillates spontaneously on both pulsations ω and $\omega_1-\omega$.

One special case is that in which $\omega_1=2\omega$, $k_1=0$. This calls to mind the classical problem of a self-excited oscillator.[5]

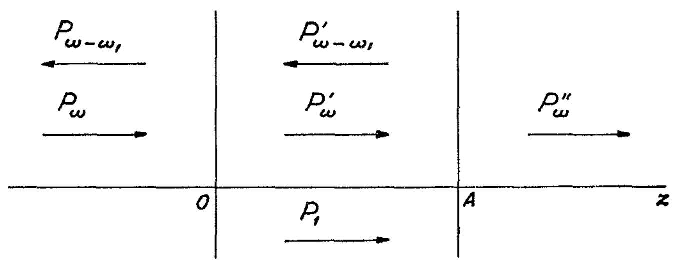

Fig. 4

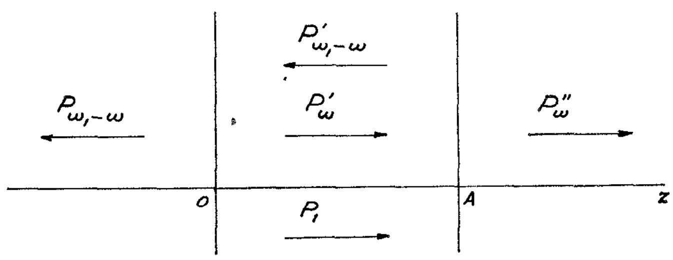

Fig. 5

[5] See paragraph 51 of [2].

Thus, in the case for which the relations

$$\begin{cases} \omega_1 > \omega \\ \dfrac{k_1}{k_0} = 2 - \dfrac{\omega_1}{\omega} \\ z_0 k_0 \dfrac{\epsilon_1}{4\epsilon_0}\sqrt{\dfrac{\omega_1}{\omega} - 1} = \dfrac{\pi}{2} + \kappa\pi \end{cases} \tag{41}$$

are satisfied, the element OA behaves as an oscillator at frequencies ω and $\omega_1 - \omega$. For given values of ω_1 and k_1, this condition can arise fortuitously, since the domain allowed to the electromagnetic wave $(\omega_1 k_0)$ is generally considerable. These conditions are very similar to the condition of oscillation of the "carcinotron" tube.

Section IV

A. Triple Root Case: Relation 1) or 3)

The following is a case of triple degeneration.

$$X_1 = \Omega; \qquad \frac{k_1}{\omega_1} = \frac{k_0}{\omega_0} \cdot$$

The three brackets of (12), (13) and (14) become zero, and coefficients a_0, a_1 and a_{-1} are of the same order. We must therefore consider (12), (13), and (14). In actual fact we shall examine not only the case where $k_1/\omega_1 = k_0/\omega$ but also neighboring cases, which will give an idea of the stability of the solution.

Let $X = 1 + \theta$; $X_1 = \Omega + \theta_1$. The parameter

$$\theta_1 = \frac{k_1}{k_0} - \frac{\omega_1}{\omega}$$

is a measurement of the difference between the phase velocity of the unperturbed wave and that of the perturbation.

Eqs. (12), (13) and (14), which are in fact the fundamental equations of the problem, are written with these new variables, assuming that θ and θ_1 are small compared to unity:

$$\frac{\theta_1 + \theta}{\Omega + 1} a_1 - \frac{\sqrt{Z}}{2} a_0 = 0 \tag{42}$$

$$\theta a_0 - \frac{\sqrt{Z}}{2}(a_{-1} + a_{+1}) = 0 \tag{43}$$

$$\frac{\theta_1 - \theta}{\Omega - 1} a_{-1} - \frac{\sqrt{Z}}{2} a_0 = 0. \tag{44}$$

Eqs. (15) or (16), obtained by éliminating a_{-1}, a_0 and a_{+1} from these three equations, is written, with these new variables:

$$Z = 2\theta \frac{\theta^2 - \theta_1^2}{\theta - \theta_1\Omega} \cdot \tag{45}$$

B. Case $\Omega = \omega_1/\omega < 1$

Fig. 6 represents the function $Z = f(\theta)$ for case $\theta_1 >$ zero. Case $\theta_1 < 0$ is deduced from the former by

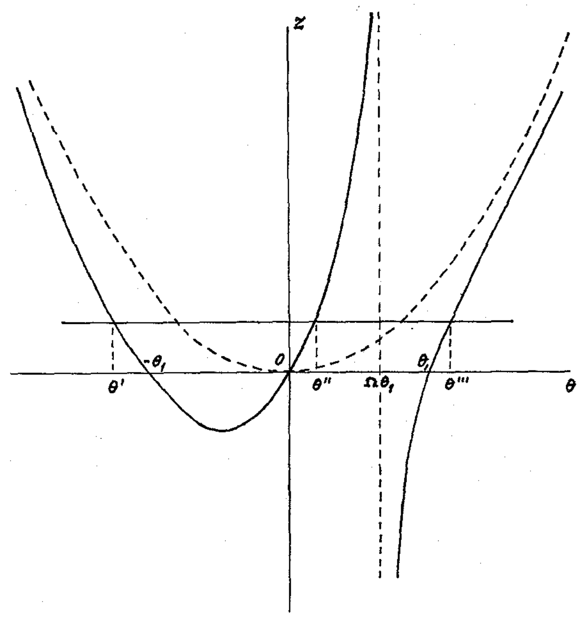

Fig. 6

changing θ_1 to $-\theta_1$ and θ to $-\theta$ (symmetry with respect to the Z axis).

For Z small and positive, there are three real roots θ', θ'' and θ''' for $Z = f(\theta)$. θ' is close to and less than $-\theta_1$; θ'' is close to zero and positive; θ''' is close to and greater than θ_1. If θ_1 tends toward zero, the solid curve blends with the parabola $Z = 2\theta^2$ shown as a dashed line in Fig. 6, and the Z axis.

If θ_1 is small, the solutions $\theta_1 \neq 0$ are of a kind differing little from the solutions $\theta_1 = 0$—a group of slow waves, a group of fast waves, and a group with velocities very close to that of the unperturbed wave.

Let us therefore examine the case $\theta_1 = 0$. The two groups of fast and slow waves have for the value of k:

$$\frac{k}{k_0} = 1 + \alpha \frac{\epsilon_1}{2\sqrt{2}\epsilon_0} \qquad \text{with } \alpha = \pm 1. \tag{46}$$

The corresponding values of a are:

$$\frac{a_{-1}}{a_0} = \frac{\alpha}{\sqrt{2}}(1 - \Omega) \tag{47}$$

$$\frac{a_{+1}}{a_0} = \frac{\alpha}{\sqrt{2}}(1 + \Omega). \tag{48}$$

For the group of waves of the same velocity as the unperturbed wave $k = k_0$, we have for the solution of the system (42), (43), and (44): $a_0 = 0$, and $a_1 + a_2 = 0$. The latter solution is of little interest since it does not agree with an incident wave of phase $\omega t - k_0 z$.

Therefore the two groups of waves described by the solutions (46), (47) and (48) must be used. All these waves are propagating in the same direction. Since their phase velocities are slightly different, they beat with one another. Actually, this produces a sinusoidal modulation of the amplitude at the various pulsations $\omega - \omega_1$; ω; $\omega + \omega_1$. The phase of this modulation is $k_0 \mu z$ with $\mu = \epsilon_1 / 2\sqrt{2}\epsilon_0$.

Writing accord at O, we have for the field expression:

$$E_x = \cos k_0\mu z \exp - j(\omega t - k_0 z) + \frac{j}{\sqrt{2}} \sin k_0 \mu z$$
$$\cdot\left[\left(1 - \frac{\omega_1}{\omega}\right) \exp - j[(\omega - \omega_1)t - (k - k_1)z]\right.$$
$$\left. + \left(1 + \frac{\omega_1}{\omega}\right) \exp - j[(\omega + \omega_1)t - (k + k_1)z]\right]. \quad (49)$$

The energy passes alternately from the vibration at frequency $\frac{\omega}{2\bar{u}}$ to the vibrations at frequencies $\frac{\omega - \omega_1}{2\bar{u}}$ and $\frac{\omega + \omega_1}{2\bar{u}}$.

C. Case $\Omega = \omega_1/\omega > 1$.

In Fig. 7 the solid curve shows the variation of $Z = f(\theta)$ for $\theta_1 > 0$. The curve $\theta_1 < 0$ is deduced from the latter by symmetry with respect to the Z axis.

If $0 < Z < \zeta$, or if $Z > \zeta'$, there are three real roots for θ. But if $\zeta > Z > \zeta'$, two of the real roots are transformed into complex roots.

Developing in the neighborhood of point A, we have:

$$X = 1 + \frac{\theta_1}{\sqrt{3}} [1 + \alpha j u], \quad (50)$$

with

$$\alpha = \pm 1; \quad u^2 = \frac{2}{3}\left(\frac{Z}{z} - 1\right); \quad \zeta = \frac{4\theta_1^2}{3(\sqrt{3}\Omega - 1)}$$
$$\Omega = \frac{\omega_1}{\omega}; \quad \theta_1 = \frac{k_1}{k_0} - \frac{\omega_1}{\omega}; \quad Z = \frac{\epsilon_1^2}{4\epsilon_0^2};$$

or

$$\frac{k}{k_0} = 1 + \frac{1}{\sqrt{3}}\left(\frac{k_1}{k_0} - \frac{\omega_1}{\omega_0}\right)\left[1 + \alpha\sqrt{\frac{2}{3}\left(\frac{Z}{\zeta} - 1\right)}^{1/2}\right]. \quad (50')$$

The value of k is complex only if $Z > \zeta$, *i.e.*, assuming that 1 is small compared to $\sqrt{3}\Omega$.

$$\frac{\epsilon_1^2}{\epsilon_0^2} > 3\frac{\omega}{\omega_1}\left(\frac{k_1}{k_0} - \frac{\omega_1}{\omega}\right)^2. \quad (51)$$

Eq. (51) gives a threshold for the appearance of exponential solutions, tied to the depth of perturbation. This threshold is lower as ω_1 becomes greater than ω. It should be noted, however, that the solution is valid only if $\omega_1/\omega_1 \# k_0/\omega_0$.

The phase velocity V is the same if $\alpha = +1$ or $\alpha = -1$; it is given by:

$$\frac{1}{V} \# \frac{1}{v_0} + \frac{1}{\sqrt{3}} \frac{\omega_1}{\omega}\left(\frac{1}{v_1} - \frac{1}{v_0}\right). \quad (52)$$

Note that if

$$\begin{cases} \theta_1 > 0 & v_1 < v_0 & v_1 < V < v_0 \\ \theta_1 < 0 & v_1 > v_0 & V < v_0 < v_1, \end{cases} \quad (53)$$

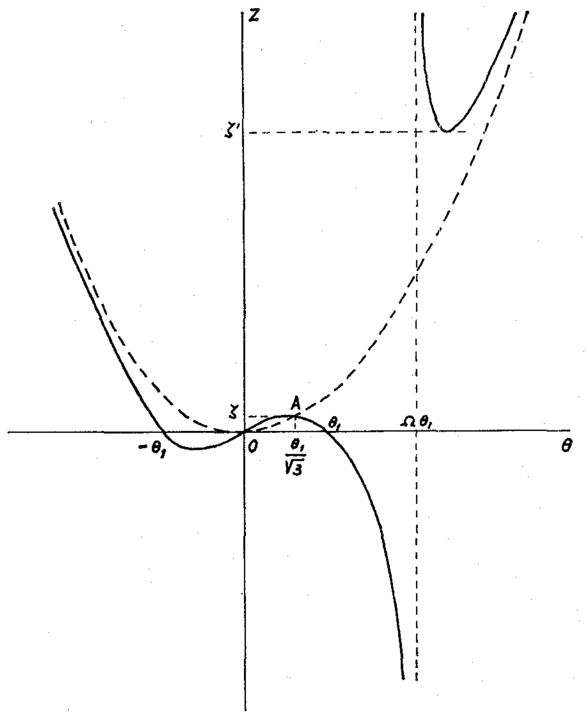

Fig. 7

the phase velocity of the exponential waves is always less than the phase velocity of the unperturbed wave, whether the perturbation phase velocity is lower or higher than the latter.

If Z is much greater than ζ, the value of k is no longer given by the approximation of (50) but is complex if $\zeta > Z > \zeta'$:

$$X = 1 + v[1 + \alpha j u]; \quad (54)$$

u and v must be calculated directly. But it can be stated that

$$\frac{\theta_1}{\sqrt{3}} < v < \frac{3\theta_1 \Omega}{2}.$$

The relations expressed in (53) are still valid, and the exponential waves are always slower than the unperturbed wave.

A group of sinusoidal waves corresponding to the real root of $Z = f(\theta)$ must be associated with the two groups of exponential waves. The latter becomes

$$X \# 1 - \theta_1 - \frac{Z}{4\theta_1}(1 + \Omega). \quad (55)$$

Verification is made of the fact that, since $\theta_1 \ll 1$, all the waves considered have positive phase velocities. The solution will therefore be obtained by bringing the fields into agreement in the plane $z = 0$. We have

$$\left[\begin{aligned} & b_1 + b_2 + b_3 = a \\ & b_1 \frac{\sqrt{3Z}}{2\theta_1} \frac{\Omega - 1}{\sqrt{3} - 1 - ju} + b_2 \frac{\sqrt{3Z}}{2\theta_1} \frac{\Omega - 1}{\sqrt{3} - 1 + ju} \\ & \qquad + b_3 \frac{\sqrt{Z}}{2\theta_1} (\Omega - 1) = 0 \\ & b_1 \frac{\sqrt{3Z}}{2\theta_1} \frac{\Omega + 1}{\sqrt{3} + 1 + ju} + b_2 \frac{\sqrt{3Z}}{2\theta_1} \frac{\Omega + 1}{\sqrt{3} + 1 - ju} \\ & \qquad - b_3 \frac{2\theta_1}{\sqrt{Z}} = 0. \end{aligned}\right. \tag{56}$$

b_1, b_2 are the arbitrary coefficients affecting the groups of exponential waves $\alpha = +1$, $\alpha = -1$, and b_3 is the arbitrary coefficient of the group of sinusoidal waves. a is the coefficient of the incident wave. The group of equations in (56) is obtained by equating the field values for pulsations ω, $\omega - \omega_1$, $\omega + \omega_1$. The system in (56) of three equations with three unknowns has, in general, only one solution.

In the case $\theta_1 > 0$ it is the group of waves corresponding to $\alpha < 0$ with coefficient b_2, which increases exponentially; in the case $\theta_1 < 0$ it is the group of waves corresponding to $\alpha > 0$ with coefficient b_1. At first sight, there is no essential difference between the two types of solutions, except for an amplitude ratio which is different in the components at the various frequencies.

Section V

A. Physical Applications

The foregoing paragraphs have established a certain number of results which can find application, in particular, in the so-called domain of "parametric amplification."

Before passing to this domain, it is useful to recall a few features and conditions of the results obtained.

a) The application of the method of first order perturbations assumes, basically, that the neglected quantities do not influence the exact solution. In the cases dealt with, this assumes that ϵ_1/ϵ_0 is small, and that the components with pulsations $\omega + n\omega_1$ with $n > 1$ are negligible. In general, this approximation seems to be reasonable physically. This means that the spectrum decreases rapidly around pulsation ω.

This may not be the case for a Bragg interference of order greater than unity, described by the relation

$$\left(\frac{k_0 + nk_1}{\omega + n\omega_1}\right)^2 = \left(\frac{k_0}{\omega}\right)^2.$$

In fact, in an actual physical problem, it is necessary to take into consideration harmonics of high order.

b) It has been implicitly assumed that extraction of energy did not modify the perturbation. This can be true only for weak signals, or when it is possible to feed the medium at all points with energy of correct phase and amplitude. Otherwise, it would be necessary to consider the fact that the electromagnetic wave at ω would have some effect on the medium itself by weakening or strengthening the perturbation. (This is similar to the action of the perturbation on the electromagnetic wave by increasing or decreasing it.)

c) The cases examined have this in common—they all relate to Bragg interference of the first order. There is a clear feeling that it is in the case of Bragg interference that the interaction of the medium on the wave is strong, for it is then that the phase conditions which make the action cumulative are best obtained. A particularly striking example is that of cases 2) and 4) (Section III) for small values of ω_1/ω.

This has led to the exploration in plane k_1/k_0, ω_1/ω of two straight lines [cases 2) and 4)] and of a region around the first bisector [cases 1) and 3) and surroundings].

In order to exhaust the problem it would, in fact, be necessary to explore the whole plane. But then the question would be raised of the validity of the solutions obtained, which, it should be remembered, are only approximations.

B. Parametric Amplification

As was stated in Section I it is possible to approach the problems of parametric amplification by separating the difficulties. A pumping energy modifies the characteristics of the medium. Knowing the modification of the medium, an action on the signal is deduced. It is then unnecessary to introduce the pumping field in the equations. This is obvious in the case of an energy of a different character—mechanical for instance. Consider the case of an electromagnetic pumping energy: that is, the same kind as the signal.

Let E_1, H_1, E, H represent the pumping field of frequency $\omega_1/2\pi$ and signal field of frequency $\omega/2\pi$. By hypothesis, the Maxwell equations (2), (3), (4) and (5) and the boundary conditions are satisfied for the pumping field E_1, H_1 alone. In particular, (4) is written

$$\nabla \times \overline{H}_1 = \frac{\partial \epsilon \overline{E}_1}{\partial t} \cdot \tag{57}$$

In (57) ϵ is a function of x, y, z and of the field E_1, H_1. In particular it can take the form described by (1).

It is now necessary to satisfy the Maxwell equations and the boundary conditions for the sum of the pumping and signal fields. But the Maxwell equations, as well as the boundary conditions, are linear. Since the pumping field already satisfies them, it is necessary and sufficient that the signal field satisfy them, provided that ϵ is formulated so that it can be determined by (57). This assumes that the addition of the signal field does not modify the value of ϵ. This condition, generally realized, can be described as an "approximation for small signals."

This being said, up to the moment and as far as we know, parametric amplification experiments have been performed only with lumped constants or with cavities. In the case of lumped constants, the circuits have always been tuned, which, as in the cavities case, is the same as repeating in time the action of the medium on the electromagnetic wave. More precisely, it is possible to pass easily from the case of a traveling wave to that of a cavity. Thus, as is well-known, in a cavity the field can generally be given the form of two traveling waves circulating in opposite directions so that the matching conditions may be obtained at the extremities. A transposition of the solutions found for traveling waves can therefore be made for cavity problems.

C.

Various analogies can be drawn from known cases of localized parametric amplification.

Up-converter: An input signal of frequency ω and "pumping" power of frequency ω_1 produce a signal of frequency $\omega+\omega_1$. The latter is amplified in power in the ratio $1+\omega_1/\omega$. This is what was found in Section III, B; in particular, see (22).

Down-converter: An input signal of frequency ω and "pumping" power of frequency ω_1 produce a signal of frequency $\omega-\omega_1$ (see Section III, D).

If $\omega L>\omega$, the resultant signal is diminished. Power loss is given by $1-\omega_1/\omega$ (see Section III, E).

If $\omega_1>\omega$, it is possible to obtain some amplification (see Section III, F). However, there is a tendency to instability; under certain conditions the system can break into oscillations, as in (41). As was already seen, this case approaches the conventional self-excited oscillator.

D.

The solution examined in Section IV should be compared to the operation of traveling wave tubes, especially in the case where $\omega_1>\omega$. (See the preceding paragraph.)

Thus, drawing an analogy between dielectric and electric current, we find that, as in the TWT case, it is in the neighborhood of equality of phase velocity of the cold wave and of the velocity of the electrons, and that it is possible to obtain exponential amplification. The amplified wave, sometimes designated "forced" wave, in TWT tubes always has a phase velocity which is less than the phase velocity of the "cold" wave [see (53)].

E.

It has been found (Section III, F) that under certain conditions described by (41) the perturbed medium may behave like an oscillator delivering energy at frequencies ω and $\omega_1-\omega$. It is interesting to point out that there is a strong analogy between the conditions in (41) and those which give the oscillating state for the UHF tube called "carcinotron" or backward wave oscillator.

VI. Conclusion

This investigation has made it possible to obtain the modes of action of a perturbation of the medium on a guided electromagnetic wave. This action is intense, especially when the so-called Bragg phase velocity conditions are obtained. It is then possible to have energy transfer into the electromagnetic wave.

In such a problem, the "boundary conditions" are as important as the solution of the propagation equation. In particular, the "accord conditions" between a perturbed and an unperturbed medium have set aside a certain number of solutions which could have been taken as actual amplifications. Such a state of affairs is found in certain plasma waves which cannot be "extracted" from the medium.

In the "approximation of small signals," it has been seen that the problem discussed is related to the problem of parametric amplification. Analogies have been found with cases of localized parametric amplification and with the conditions of traveling wave tubes and carcinotrons.

We do not claim to have completely dealt with parametric amplification for traveling waves, but it seems that this way of treating the problem may bring out the greatest number of related physical concepts. It appears that the complete solution of practical cases should be undertaken in accordance with the proposed scheme which makes it possible to separate the difficulties—action of the "pumping energy" on the medium, and action of the perturbed medium on the signal.

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